

Some Thoughts about Volatility and Derivatives

Summary First Part

The objective of this article is to form a theoretical reference which, when combined with a computational process, will permit a numerical handling of the Black and Scholes equation. This theoretical mark will facilitate the measurement of the implied volatility with sufficient precision to allow Delta to be significant. Where Delta is the probability of execution of an asset. Given the initial result, the second desired objective is to use the prior knowledge to build a posteriori probability distribution based on the Bayesian type of inference. Furthermore, this mark can be used to stimulate a correction in the Black and Scholes Model. Once we reach these two objectives, we will have a model which is resilient and has a good foundation to manipulate financial strategies. Even though financial strategies are common in capital markets, with this procedure, they will have the possibility to maximize a desired profit level.

Key words: Black & Scholes, Bayesian Inference, Options, Volatility.

Introduction

According to STEWART (2015), there are 17 equations that changed the world. The first is the Pythagorean Theorem and the last is the Black and Scholes Model. Much has been said about the first and more will be said about the second. As a matter of fact, the inappropriate application of the second have already bankrupted, at least, one investment fund.

Nevertheless, we know that investment and crisis are important parts of the human history. Paying attention to history can be a picturesque way to look at the future or to predict events which may happen with certain probability. For instance, the Tulip Bubble, during the XVII century in Holland, ended in 1637 with a confidence loss in the financial system of the time. The loss of trust in the system broke precipitously countless business in Europe, and left countless businessmen not knowing what to do. Many other economic crisis followed the famous Tulip Bubble in Holland, and there are many and diverse reasons for them to happen. And, for additional information, I will mention some of these crises. The most famous is the crash of the American Stock Market in 1929 and led the world into a depression. However, there were others, like the hike in the price of petroleum in 1970's; the debt crisis of the developing countries in 1981; the crisis of the Mexican peso in 1994; the financial crisis of east Asia in 1990s, and finally, that of Russia

and of Latin America in 1998. With a greater or a lesser intensity, the lost of trust was and will be present in these crises as an invariant factor. Even recently, the crisis of the sub prime in the american market in 2007/2008 as very similar to the crisis that broke the Dutch market in 1637. Again, it shook the trust in the markets; and it practically lowered to zero the ratings of many financial institutions in the United States and Europe.

What do we learn from these crisis so similar and so far apart in time? We have learned little or very little from the similarity, but what we have learned is sufficient to imagine that other crises will happen in similar fashion. The crisis of trust have to be relevant to the real world as posted by Lacan. However, as human beings, we have difficulties to symbolize it and to make the connection between the REAL and the imaginary. It is true that the crises are always present in the history of the world.

However, it doesn't mean that the path of progress would be a repetition of mistakes or a stagnation of financial processes. The attempt to understand, to imagine or to symbolize any crisis may spring from situations which are singular or even paradoxical. Thus, walking slowly through the time horizon of the financial evolution, we meet Robert Brown, a British biologist in the XIX century. He discovered an incredible parallel between the grains of polens moving over water with stochastic variations in the price of an asset. Based on the movement observed by Brown, several researchers have generate price models of assets and their respective derivatives.

For example, as a result of Brown's work, in 1973, the American researchers, Fisher Black and Miron Scholes (BLACK, 1973) got high recognition for an article describing their pricing model for premium of buying and selling stock options. A few years later, in 1977, the researcher Scholes and Merton (Fisher had died) received the Nobel Prize for economics for their work. Their equation is known today as the Black and Scholes Model or MBS:

$$c = S \cdot N(d1) - Ke^{-rT} N(d2)$$

$$d1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d2 = \frac{\ln\left(\frac{S}{K}\right) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d1 - \sigma\sqrt{T}$$

where,

$$N'(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

and,

- c : Price of an option Call
- r : interest rate free of risk
- σ : volatility
- K : Strike price
- T : Exercise time for the option
- S : the underlying price of the asset.

Given that much has been written about the Brownian movement of particles and about the Black and Scholes equation, it is unnecessary to make here a historical or theoretical revision of the many work already done in this line of research. However, there are two articles which are fundamental for our line of inquiry. First, it is important to mention MANASTER (1982), who laid the ground to ease the application of the Newton-Raphson method to measure the implied volatility. Manaster method permitted automated calculations by using written routines, for instance, base on the R language. By following his foot steps, we should also mention the work of DUMAS (1998). Dumas work is a proposal to estimate the implied volatility via a linear multiple regression. It is an important advancement because he uses a part of the existing covariables already in the Black and Scholes models.

The article of Manaster and Dumas are complimentary. Taking them together, they allow us to manipulate numerically the MBS model by using computational methods. This allows us to infer and to explain the formation of price in the derivative chain. Basically, this article works with the ideas and the theories of the two researchers just mentioned in the previous paragraph. Our initial idea was to build a small theoretical criticism of the model proposed by DUMAS. But the criticism became a refinement of the linear model and, consequently, an improvement in the quality of the estimate of the implied volatility of the derivative. The other leg of our work is much ambitious; its is Bayesian type proposal which will permit a probabilistic correction of the MBS. The combined effort to understand DUMAS, and to built a slight alternative for the proposal of Manaster and Dumas, it is necessary to manipulate algebraically the equations of the MBS model.

It is important to note that all the covariables of the Black and Scholes model, with exception of implied volatility - σ_{real} – are known and that they can be easily obtained

in the capital markets. Thus, to turn the exercise proficuous it would be necessary just to rewrite the equation

$$c = S \cdot N\left(\frac{\ln\left(\frac{S}{K}\right) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}N\left(\frac{\ln\left(\frac{S}{K}\right) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}\right)$$

isolating the only unknown covariable, σ_{real} . Additionally, it would be necessary to determine a function such as:

$$\sigma_{\text{real}} = f(c_t, r, K, T, S_t).$$

Unfortunately, there isn't an analytical solution for this equation. Consequently, the result needs to be reached numerically by approximation. Lemgruber (1995) guarantees a unicity of the implied volatility for the support given by the covariables of the MBS. He proposes a very intuitive, almost artisan, method to calculate the same variables. Even though, these measurements are functional, they render this methodology impractical. This is especially true when one needs lots of financial calculations within a short time span. This is crucial, especially when these calculations cannot be automated easily. Notwithstanding the difficulties, Haug (2006) and Natenberg (1994) show an adaptation for the Newton-Raphson algorithm which has good computational characteristics:

$$\sigma_i = \sigma_{i-1} - \frac{c_{\sigma_{i-1}} - c_m}{K_{i-1}}$$

where,

- σ_i : is the implied volatility;
- $c_{\sigma_{i-1}}$: is the premium of the option call at the moment σ_{i-1}
- c_m : is the market value of the buying option;
- K_{i-1} : is “Vega” of the buying option at the time (i-1).

With the Newton and Raphson method well solved, Manaster measured the seed, σ_1 , in such way that all computational effort can be reduced to:

$$\sigma_1 = \sqrt{\left|\log\left(\frac{S}{K}\right) + rt\right| \frac{2}{t}}$$

where,

- t : is the time to exercise an option contract.

With these measurements in hand, it possible to get the following set of data

$$(\sigma_i, c_i, K, S_i, r, t_i).$$

And with this set, Dumas realized that he could build a linear polynomial model to estimate, with some precision, the implied volatility of small variations around the values obtained from the market, that is to say:

$$\sigma_{i+1} = g(K, S_i, r, t_i)$$

More specifically,

$$\sigma_n = \beta_0 + \beta_1 K + \beta_2 K^2 + \beta_3 t + \beta_4 t^2 + \beta_5 t K$$

is a simple equation easy to set up and yet it can estimate the implied volatility to be used in the MBS, offering thus a forecast of the option price. Even more important is the consequences of the model proposed by Dumas; it results in a very dependable Delta.

Some critique to Dumas Model

To avoid spurious correlations, Dumas was very careful to use only part of the parameters used in the MBS model. furthermore, Dumas avoid additional informations of fundamentalistic nature or from graphic analysis -- such as moving averages and oscillators which are well known by financial analysts.

It is important to remember that Dumas made use of linear models, common at the time he wrote the article. Nowadays, the generalized linear models or GLM by itself produce more efficient models and results. Consequently, more efficient models produce more dependable Delta, which in turn strengthens the Bayesian inference applied to MBS as a correction factor.

Here is a list of the themes which will be explored as a small critique of the model developed by Dumas:

- Razões para o GLM
- Razões do Mercado
- Correção Bayesiana

A GLM to explain Implied Volatility

When Dumas decide to use a polynomial regression to estimate the implied variance and to measure the just premium of a derivative, he, automatically, elected the Normal distribution model. In doing so, he accepted all theoretical implications of the this model. Therefore, it is important to note that, according to PAULA (2010), the variance function of a Normal curve is constant, that is to say,

$$V(\mu) = 1$$

The consequences of this condition are very important. The experience coming from the real world show that, it is an impossible supposition.

Financial analysts, with knowledge and experience in the capital markets, know that positive news affect the price of the assets and of the derivatives. They also know that the influence of the positives facts are different from those negatives happenings. Furthermore, they know that the duration of the negative influence last longer than those coming from positive forces.

The heteroscedasticity of the variance of errors affects the averages of the observations. As such, they should be taken into account when building the model. Obviously, this behavior changes the variance of phenomenon under study. The probability that a function, with a constant variance, could explain the changes in a phenomenon under study is very low. Consequently, it would be interesting to use different probability distribution that could represent the variance of errors much better. This is the case of General Linear Models (GLM).

Even though the GLM model offers several distribution of probability, two of them are specially important, namely Gamma and the Inverse Normal. The Gamma distribution has the following function of variance:

$$V(\mu) = \mu^2$$

which presents much more interesting possibilities than the Normal distribution. However, the inverse Normal distribution depicts a cubic relation between the average and the variance,

$$V(\mu) = \mu^3.$$

This relationship allows the model to explain more convincingly the heteroscedasticity. Perhaps, this explanation is much more coherent with the positive and negative phenomena coming from the capital markets.

Beyond these questions pertinent to the variance function, it is important also to consider the care that Dumas took to avoid structuring his regression on covariables. By avoiding covariables he protected his model from becoming a spurious relationship. Therefore, some important recommendations are called for to help in the construction of a more robust model. Market analysts know that:

- The force of positive and negative facts work differently overtime. This timely uneven effect suggests that, there might be an autoregressive structure

- There is a variety of indicators and oscillators which are largely used in trend, in risk and in variance analyses;
- The analysis of graphic patterns is a common instrument. It reveals known behavior which, to a point, is dependable;
- Fundamental Indicators and indices are important to chose an asset, but they do not have the necessary speed to explain the volatility of the respectives derivatives.

Ultimately, Dumas model should maintain its polynomial structure, but should include some of the covariables suggested before:

- $o_1 = S_{t-1}$;
- $o_2 = K$;
- $o_3 = C_{t-1}$;
- $o_4 = t$.

Furthermore, we can add, to this set of covariables, the following relationships in building the polynomial structure adopted by Dumas. This way the model would be much more inclusive:

- $p_1 = S_{t-1} \times K$
- $p_2 = S_{t-1} \times C_{t-1}$
- $p_3 = S_{t-1} \times t$
- $p_4 = K \times C_{t-1}$
- $p_5 = K \times t$
- $p_6 = C_{t-1} \times t$

and

- $q_1 = S_{t-1}^2$
- $q_2 = K^2$
- $q_3 = C_{t-1}^2$
- $q_4 = t^2$

Also, it would be interesting to add some indicators to these covariables. They will help in the composition of the analysis of the graphic indicators - we call them noise:

- $r_1 = \log(\text{MedDr}/\text{MedAt})$
- $r_2 = \log(\text{QttDr}/\text{QttAt})$

where,

- MedAt: average of the asset at time, t;
- MedDr: average of the derivative at time, t;
- QttAt: quantity of assets traded at time, t;
- QttDr: quantity of derivatives traded at time, t.

Finally, taking an identity connection, the new model could be rewritten as:

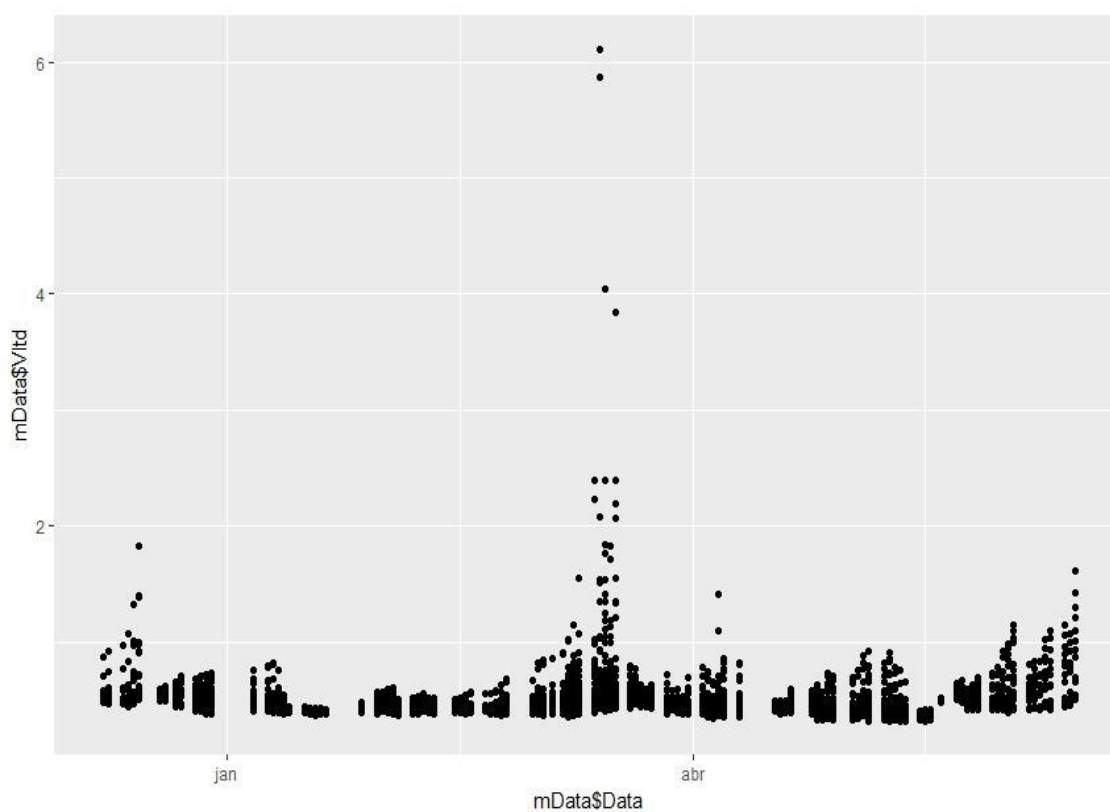
$$\sigma_{t+1} = \sum_{i=1}^6 \beta_{ip_i} + \sum_{i=1}^4 \beta_{io_i} + \sum_{i=1}^4 \beta_{iq_i} + \sum_{i=1}^2 \beta_{ir_i}$$

whereas the GLM model still permits some additional refinements, such as, the use of the logarithm and the inverse connections.

The possibility to estimate the implied volatility and explain it through the use of a set of covariables is an interesting aspect of the use of linear models instead of time series. As I was writing this article, I also collected data about of high liquidity assets in the Bovespa index, such as: – VALE5, PETR4, ITUB4 e BBAS3. I also estimated more than 51 regression. From the 18 covariables used during the modeling, only 5 of them, namely - p_1 , p_2 , p_4 , q_3 e q_4 – were significatives in all the adjusted models.

Accepting that this set of covariables in the model, when studied in isolation might be different; we could suppose that each assets or group of assets not only behave differently but also has their own formation laws. For illustration purpose only, the next two tables depict some important results. The first is the behavior of the volatility of Petrobras stocks (PETR4) within the period under studied.

Figure 1: Implied Volatility of PETR4 in 2017



in second place we have the results obtained with the MGL applied on the same asset using data collected during 3 months under analysis:

Table 1: Estimates of the Implied Volatility with the proposed model

Option	K	c_t	t	σ_t	Call BS	Δ
PETRH11	11.00	2.33	15	0.342	2.36	0.991
PETRH91	11.50	1.86	15	0.341	1.88	0.968
PETRH46	11.75	1.63	15	0.339	1.64	0.946
PETRH57	12.00	1.41	15	0.339	1.41	0.911
PETRH42	12.50	0.96	15	0.323	0.98	0.815
PETRH13	13.00	0.59	15	0.318	0.62	0.657
PETRH48	13.25	0.44	16	0.306	0.47	0.565
PETRH43	13.50	0.33	15	0.321	0.36	0.468
PETRH73	13.75	0.23	16	0.307	0.26	0.376
PETRH14	14.00	0.16	15	0.320	0.18	0.292
PETRH68	14.25	0.12	15	0.328	0.13	0.226
PETRH45	14.50	0.08	16	0.323	0.10	0.172
PETRH65	15.00	0.05	16	0.342	0.05	0.099
PETRH75	15.25	0.02	16	0.339	0.03	0.068

In this example, when we adjust the data, we may observe the closeness between the premium paid by the market (c_t) and the just premium, measured with the MBS (CallBS). After we observe these preliminary results, we can see that the market uses the MBS only to gauge the price of an option. However, the Implied Volatility, when well modeled, as proposed, present a Δ – which is the probability that an option will executed – this become a fundamental role in all investment strategies.

A Bayesian Strategy

What is the bet strategy to trade? This is an intriguing question! It is always present in the mind of those laboring in the financial sector. Even though, almost all investors assures us to trade with the best strategy; there is none. And there never will be a definitive answer this question.

However, when we trade with covered calls to hedge our market operations, taking advantage of good strikes with high premium, it becomes one of the best strategies. This strategy is very strong when it is associated with a high probability of strikes deep in the money. That is to say, this strategy is simple, but not ingenuous. We could resume it, as how we can maximize the premiums when we write a call. With this strategy, in the worst case scenario, we want to be exercised in the money or stroked out at the end of the contract.

As we already alluded, the advantage of the proposed model is to estimate with plenty of precision the implied volatility of a derivative. The advantage coming out of the implicit value of the model allow us to raise a few but important assumptions within the sphere of the financial market.

The derivative of the estimated premium in relation to the price, according to Wilmott (2006) is:

$$\Delta = \frac{\partial c}{\partial S} = \int_{-\infty}^{\frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} dx$$

and it gives the probability of the derivative of the exercise $-\Delta$. This probability is interesting because it deals with a priori probability; that is to say, an information that the investor has before the expiration of the option contract. We should note that, in this case, the exercise of the option will be in the money, when S and K have values very close to each other. Therefore we can state that, in light of a possible small error,

$$error = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \sim 0$$

that when the price of the derivative is equal to the execution price evaluated the $\ln(S/K)$ will be equal to zero. The probability of the exercise in the money can be estimated through an obvious mark, equivalent to the evaluated probability in the quartile Q_{50} of the function density probability described by the derivative that originates Δ .

Knowing this probability before hand, it is possible to build an experiment to corroborate empirically with information given, to combine them and to offer a new perspective coming from the market. In the end, this is the strength of Bayesian Inference.

Construction of a Predictive Distribution

With a Bayesian Inference is possible to build a posterior distribution, made up of a probabilistic combination which mixes of information coming from previous knowledge – a prior distribution – with empirical information collected in the market, in the case of a likelihood.

We define Θ : a random variable, to represent the probability of exercise and $X|\theta$: a random sample taken from a population with an exercise probability of Θ . In this context a posterior distribution will be proportional to the product of prior likelihood, or simply:

$$\pi(\theta|x) \propto l(\theta;x) \cdot h(\theta) .$$

It should be clear that several distributions can be used to arrange this construction, however the probability distribution better indicated to represent the information in question would be the Beta (a,b) distribution which is well known and studied. However, due to topological and numeric problems the Beta(.) distribution rendered unsuitable for some samples gathered during the experiment: more specifically it showed unsuitable to deal with the price formation of the derivatives of some assets studied. In light of this unsuccessful proceeding one should abandon the Beta(.) distribution and work with a Normal(.) Distribution. This would be the best solution of this impasse.

Supposing that both random variables - $\Theta \in X|\theta$ – are normally distributed, Ehlers (2011) demonstrated that:

$$\text{Likelihood: } X|\theta \sim N(\theta, \sigma)$$

$$\text{Prior: } \theta \sim N(\mu_0, \tau_0)$$

$$\text{Posterior: } \theta|X \sim N(\mu_1, \tau(\sigma_i)),$$

That is to say, given that the variance σ is known in its likelihood, a posteriori distribution would be a Normal Distribution whose variance τ is function of σ . The relation between variance and volatility is also known. Thus, given that the variance is a temporal composition of volatility, it is possible to build a posteriori distribution for each asset. That is, build a family of predictive distribution the derivative volatility of each asset evaluated. Even though, this approach is very useful and interesting, it also rendered inappropriate. The problem surges when the size of the sample increases. In this case, a family of normal distributions seems to converge to only one predictive distribution. This might be due to the Central Limit Theorem.

The volatility is no longer the only representative of the probability of execution, and Δ beginning to have a role in the predictive distribution. The utilization of the volatility and of Δ , at the same time, use the same sample more than once. This can induce to an undesirable mistake. The solution is to abandon the idea of posteriori families and conjugated distribution to work directly with the exercise probability, and numerically simulate a predictive distribution.

Several distribution can be used. However, the probability density function of Gompertz is particularly interesting because it deals with the survival curve and actuarial information. A philosophic parallel could easily be drawn between the life time of certain option and the execution, or not, of the respective contract.

Thus defining:

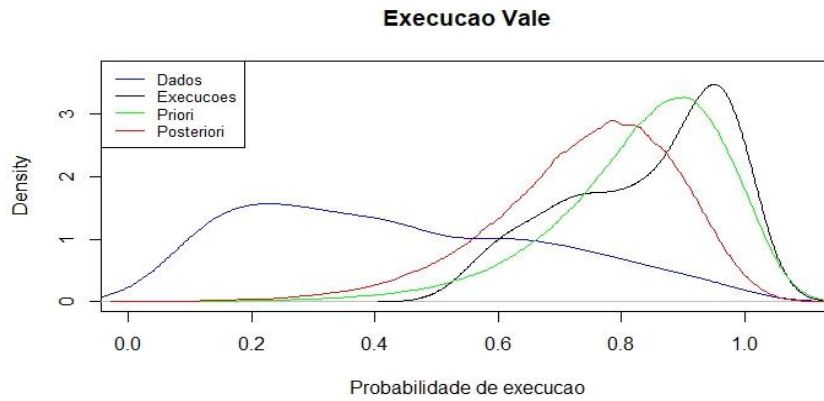
Likelihood: $X|\theta \sim \text{Gompertz}(a, b)$

Priori: $\theta \sim \text{Gompertz}(c, d)$

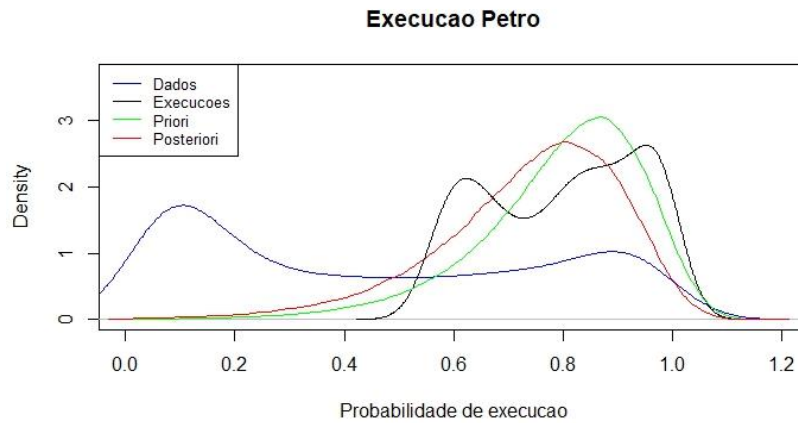
Posteriori: $\theta|X \sim \text{unknown}$

where a posteriori distribution doesn't have its primary defined and should be simulated numerically using the Monte Carlo Method with the Marcov Chains – MCMC. This simulation can be executed with the method of the package mcmcPack used for R . Some results can be observed as follows:

Vale Execution



Petro Execution



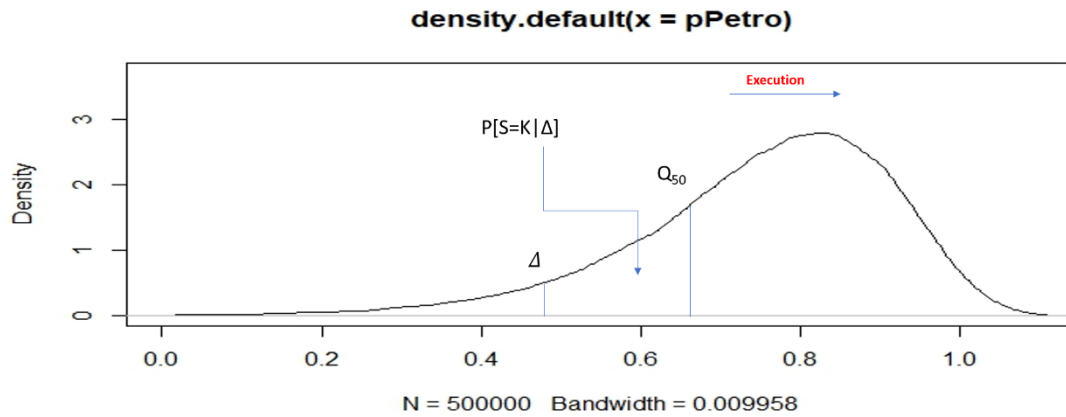
A numeric integration, through simulations is simple and intuitive, one need just to calculate the execution probability in the money conditioned by Δ . Assuming $\Delta < Q_{50}$, we have (EHLERS, 2011):

$$P[S=K|\Delta] = \int_{\Delta}^{Q_{50}} p(\theta|x) d\theta = E_{\theta|x}[\theta < Q_{50}] - E_{\theta|x}[\theta < \Delta]$$

The probability of execution in the money may also be calculated directly with information simulated from a posteriori distribution. That is to say:

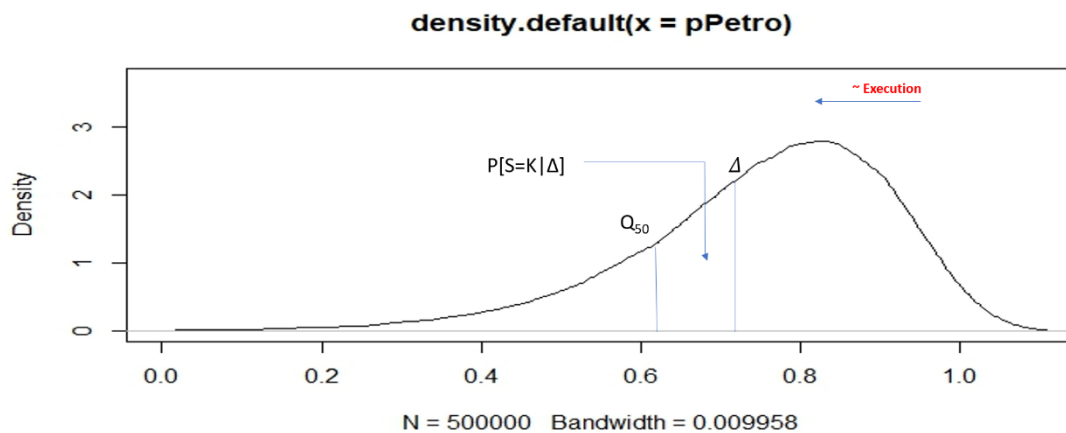
$$P[S=K] = P[S=K|\Delta] \cdot P[\Delta] = P[S=K|\Delta] \cdot E_{\theta|X}[\theta < \Delta]$$

In an inverse situation, when $\Delta > Q_{50}$, the calculation of the probability is similar. However, we should consider that the execution in the money takes place in the opposite direction as we can observe in the figures below:



Applying an analogous reasoning, it send us to an idea that the contract returns to the condition of execution in the money. That is:

$$P[S=K] = P[S=K|\Delta] \cdot P[\Delta] = P[S=K|\Delta] \cdot (1 - E\theta|X[\theta < \Delta])$$



Using the probabilities defined before, we obtain the following results for the stocks of the Bank of Brazil:

	Opção	Execução	Prêmio	Dias Úteis	VLTD	Preço Call	Δ	$P[S=K \Delta]$	$P[S=K]$
	BBASK5	35.57	1.98	30	0.323	2.02	0.60	0.53	0.32
	BBASK6	36.57	1.43	30	0.314	1.47	0.50	0.65	0.33
	BBASK7	37.57	0.97	30	0.296	0.98	0.40	0.70	0.28
	BBASK14	39.82	0.4	30	0.287	0.37	0.19	0.71	0.14
	BBASJ33	33.07	3.15	7	0.409	3.14	0.91	0.27	0.03
	BBASJ93	33.57	2.83	7	0.450	2.76	0.84	0.20	0.03
	BBASJ34	34.07	2.21	7	0.349	2.21	0.85	0.21	0.03
	BBASJ94	34.57	1.73	7	0.313	1.75	0.81	0.10	0.02
	BBASJ5	35.57	1.04	7	0.300	1.02	0.63	0.47	0.30
	BBASJ36	36.07	0.66	7	0.273	0.68	0.52	0.64	0.33
	BBASJ6	36.57	0.45	7	0.276	0.46	0.40	0.70	0.28

Conclusion

It is curious to note that, inclusion of information in a co-variable format, in the model proposed by Dumas, increases the entropy of the volatility. However, it is this increase that permits a construction of a more robust model, which has more efficient controls of the phenomenon under study. This can be seen through the precision found in the prediction of the premiums estimated by the MBL.

Although, this precision can be, in part, evaluated through the construction of a predictive distribution; the Bayesian inference allow us to connect historical data with more recent empiric information. It offers, not only a correction of the probability of the experiment, but also a measure of the unknown probability until that date. In the case:

$$P[S=K]$$

and

$$P[S=K|\Delta]$$

These corrections produce conditions to formulate strategy with a new level of confidentiality. It also increases the distance between the emotional reactions and operations which use the techniques proposed in this article.

As a reminder, as a last suggestion; it is important to make an analysis of the Principal Parts or similar components in order to evaluate Dumas concerns with the utilization of spurious co-variables. An inherent advantage, when some dimensions of the regression model are determined, lies on the fact that permits us, in thesis, to follow the macroeconomic variables and to improve the precision of the model.

Bibliography

1. ASSAF NETO, A. (2009). Finanças corporativas e valor. 4 ed. São Paulo: Atlas.
2. AZEVEDO, H. (2010). Investimentos no mercado de opções sobre ações no Brasil: teoria e prática. Rio de Janeiro: Elsevier.
3. BERKOWITZ, J. (2010). On Justifications for the ad hoc Black-Scholes Method of Option Pricing, *Studies in Nonlinear Dynamics & Econometrics*, Vol. 14: No. 1, Article 4.
4. BLACK, F.; Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81 (3): 637-654.
5. BLACK, F. (1976). Studies of Stock Price Volatility Changes. *Proceedings of the Meetings of the Business and Economics Statistics Section, American Statistical Association*, pp. 177-181.
6. BOLLERSLEV, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, vol. 31, issue 3, pages 307-327.
7. COSTA, C. L. (1998). Opções: operando a volatilidade. São Paulo: Bolsa de Mercadorias & Futuros.
8. COX, J.; RUBINSTEIN, M. (1985). *Options Markets*, Prentice-Hall, New Jersey.
9. DUMAS, B.; FLEMING, J.; WHALEY, R. (1998). Implied Volatility Functions: Empirical Tests, *Journal of Finance*, 53, 2059-2106.
10. EHLERS, R.S. Introdução à Inferência Bayesiana. Paraná: Departamento de Estatística, 2005. Disponível em: <http://www.icmc.usp.br/~ehlers/bayes>
11. ENGLE, R. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of Variance of United Kingdom Inflation, *Econometrica*, 50:987-1008.
12. FELLER W. (1970). *An introduction to Probability Theory and its Applications*. Princeton University. Volume II.
13. FERREIRA, L. (2009). Mercado de opções: a estratégia vencedora. São Paulo: Saraiva.
14. FRIEDMAN, M. (1953). *The Methodology of Positive Economics*. Essays in Positive Economics. Chicago: University of Chicago Press.

15. HAUG, E. (2006). The Complete Guide to Option Pricing Formulas, 2nd edition. New York: McGraw-Hill
16. HULL, J. (2016). Options, Futures, and Other Derivatives. 9th Ed. Pearson Prentice-Hall.
17. LEMGRUBER, E.F. (1995). Avaliação de Contratos de Opções. Ed. BM&F.
18. LIMA, I.; LIMA, G.; PIMENTEL, R. Org. (2007). Curso de mercado financeiro: tópicos especiais. 1 ed. 2. reimpr. São Paulo: Atlas.
19. MANASTER, S.; KOEHLER, G. (1982). The Calculation of Implied Variances from the Black-Scholes Model: A Note. Journal of Finance, 37, No. 1, pp. 227-230.
20. MARQUES, F.T. (2008). Estimação do Value at Risk via enfoque Bayesiano. 58f. Dissertação (mestrado). São Paulo. UFSCAR.
21. MERTON, R. (1973). Theory of Rational Option Pricing. Bell Journal of Economics and Management Science 4 (1): 141-183.
22. MORETTIN, P. (2008). Econometria Financeira - um curso em séries temporais financeiras. São Paulo: Editora Edgard Blucher.
23. NATENBERG, S. (1994). Option Volatility & Pricing: Advanced Trading Strategies and Techniques. McGraw-Hill; Updated edition.
24. NELSON, D. (1991). Conditional heteroscedasticity in asset returns. Econometrica, v. 59, n.2, p. 347-370.
25. PAULA, G.A. (2010). Modelos de Regressão com apoio computacional. USP. Disponível em: <http://www.ime.usp.br/~giapaula/mlgs.html>
26. R DEVELOPMENT CORE TEAM. (2011). A Language and Environment. R Foundation for Statistical Computing. Vienna, Austria.
27. STEWART, I. (2015). 17 Equations That Changed The World. Zahar.
28. Wilmott, P. (2006). Quantitative Finance. Ed. John Wiley & Sons.